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Noether's theorem and exact invariants for time-dependent systems

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Abstract. We apply Noether's theorem to time-dependent systems to generate an infinite number of new invariants. These invariants have proved useful in quantising time-dependent systems, playing, in effect, the role of the Hamiltonian for these systems.

1. Introduction

Exact invariants for time-dependent systems are decisive for investigating the physical properties of these systems. The simplest example illustrating this is the time-dependent harmonic oscillator, described by the equation

$$\ddot{\rho}+\omega^2(t)\rho=0. \tag{1.1}$$

Lewis (1968) proved that the quantity

$$I=\tfrac{1}{2}[(\dot{x}\rho-\dot{\rho}x)^2+(\rho/x)^2] \tag{1.2}$$

is an exact invariant for the time-dependent oscillator. Here x satisfies the auxiliary equation

$$\ddot{x}+\omega^2(t)x=1/x^3. \tag{1.3}$$

Lewis derived this result by applying the asymptotic theory of Kruskal (1962) in closed form, obtaining (1.2) and (1.3) starting from (1.1).

The invariant I was used by Lewis and Riesenfeld (1969) to construct an exact quantum theory of the time-dependent oscillator. This same invariant was used by Khandekar and Lawande (1975) who derived an expression for the Feynman propagator in terms of the eigenfunctions of I. In many ways the invariant I takes over the central role played by the Hamiltonian for time-independent systems.

Lutzky (1978) derived the invariant (1.2) and the auxiliary equation (1.3) by a straightforward application of Noether's theorem to the Lagrangian

$$L=\tfrac{1}{2}(\dot{\rho}^2-\omega^2(t)\rho^2). \tag{1.4}$$

Ray and Reid (1979) applied Noether's theorem to the Lagrangian

$$L=\tfrac{1}{2}(\dot{\rho}^2-\omega^2(t)\rho^2+2\textstyle\sum G_i(t)F_i(\rho)), \tag{1.5}$$

thus extending the results of Lutzky.

Sarlet (1978) studied, in detail, Kruskal's method of (exact) adiabatic invariants, and developed equations which, when satisfied, assured one that the method could be applied in closed form. Sarlet then determined the invariant by solving the Hamilton–Jacobi equation. Sarlet's goal was the determination of all time-dependent Hamiltonians, with one degree of freedom, for which Kruskal's method yields an exact invariant, in a way similar to Lewis's derivation of (1.2). As examples Sarlet discussed a class of polynomial Hamiltonians for which Kruskal's method leads to exact invariants.

In this paper we apply Noether's theorem to a class of Lagrangians which contain Sarlet's polynomial Lagrangians as special cases. We present not only a generalisation of Sarlet's results, but a simpler derivation using Noether's theorem.

In § 2 we present the Lagrangian we use and an outline of Noether's theorem. In § 3 we give our main results along with some explicit examples. Finally in § 4 we present our conclusions.

2. Lagrangian and Noether's theorem

The polynomial Lagrangians considered by Sarlet (1978) have the form

$$L=(1/2a)\rho^k\dot{\rho}^2+b\rho^l\dot{\rho}+c\rho^m+d\rho^n, \tag{2.1}$$

where a, b, c, d are functions of time and k, l, m, n are constants. In order to obtain Sarlet's Hamiltonians exactly one must make some reduction in the generality of (2.1) by further specifying some of the functions a, b, c, d and constants k, l, m, n. Instead of dealing with the Lagrangian (2.1) we consider the more general Lagrangian

$$L=(1/2a(t))\rho^k\dot{\rho}^2+b(t)\rho^l\dot{\rho}+\sum_i G_i(t)F_i(\rho), \tag{2.2}$$

where a, b, G_i, F_i are arbitrary functions of their arguments.

The equations of motion for the system follow from Hamilton's principle of extremising the action

$$A=\int L\,\mathrm{d}t. \tag{2.3}$$

Performing a transformation to a new time variable $\bar{t}$ defined by

$$\bar{t}=\int^t a(t)\,\mathrm{d}t,$$

the action becomes

$$A=\int \bar{L}\,\mathrm{d}\bar{t}. \tag{2.4}$$

The new Lagrangian $\bar{L}$ has the same form as (2.2) with $a=1$ and new functions $(\bar{b}, \bar{G}_i)$. In what follows we drop the bar and write

$$L=\tfrac{1}{2}\rho^k\dot{\rho}^2+b\rho^l\dot{\rho}+\sum_i G_iF_i. \tag{2.5}$$

The equation of motion for this Lagrangian is

$$\ddot{\rho}+\dot{b}\rho^{l-k}+(k/2\rho)\dot{\rho}^2-\rho^{-k}\sum_i G_iF_i'=0. \tag{2.6}$$

This Lagrangian is the one we employ in this paper.

We next outline Lutzky's formulation of Noether's theorem. A symmetry transformation for a system is described by the group operator

$$X = \xi(\rho, t)(\partial/\partial t) + \eta(\rho, t)(\partial/\partial\rho). \tag{2.7}$$

If (2.7) is a symmetry transformation for the system, then the combination of terms

$$\xi(\partial L/\partial t) + \eta(\partial L/\partial\rho) + (\dot{\eta} - \dot{\rho}\dot{\xi})(\partial L/\partial\dot{\rho}) + \dot{\xi}L \tag{2.8}$$

is a total time derivative of a function $f(\rho, t)$, i.e.

$$\xi(\partial L/\partial t) + \eta(\partial L/\partial\rho) + (\dot{\eta} - \dot{\rho}\dot{\xi})(\partial L/\partial\dot{\rho}) + \dot{\xi}L = \dot{f}. \tag{2.9}$$

It follows that a constant of the motion for the system is

$$I = (\xi\dot{\rho} - \eta)(\partial L/\partial\dot{\rho}) - \xi L + f. \tag{2.10}$$

In the usual applications the constants (2.10) correspond to conservation of energy, momentum, etc. For example, conservation of energy, for an isolated system, is associated with ξ = constant, $\eta = 0$, $f = 0$, in which case I is the constant energy of the system. In the present problem, however, the Lagrangian (2.5) does not possess any obvious symmetry, and we are, in fact, applying Noether's theorem to determine if it allows any symmetry transformations at all.

3. Examples

We apply Noether's theorem to the Lagrangian (2.5). The idea is to substitute (2.5) into (2.9) and equate powers of $\dot{\rho}$ and ρ, since the equation must be identically satisfied. Doing this the $\dot{\rho}^3$ terms yield

$$\xi' = \partial\xi/\partial\rho = 0, \qquad \xi = \xi(t). \tag{3.1}$$

The $\dot{\rho}^2$ terms then give

$$\eta = \rho\dot{\xi}/(k+2). \tag{3.2}$$

The $\dot{\rho}$ terms require

$$f = [\xi\dot{b}/(l+1) + b\dot{\xi}/(k+2)]\rho^{l+1} + \ddot{\xi}\rho^{k+2}/(k+2)^2. \tag{3.3}$$

Finally the terms independent of $\dot{\rho}$ yield the equation

$$\xi \sum_i \dot{G}_i F_i + \dot{\xi} \sum_i G_i F_i + \dot{\xi} \sum_i G_i \rho F_i'/(k+2)$$
$$= \{\xi\ddot{b}/(l+1) + \dot{\xi}\dot{b}[1/(l+1) + 1/(k+2)]\}\rho^{l+1} + \dddot{\xi}\rho^{k+2}/(k+2)^2. \tag{3.4}$$

If this equation is satisfied, then we calculate the invariant I using (2.10). In order not to burden the reader with too much tedious discussion, we concentrate on a few special cases of solving (3.4).

Equation (3.4) must be satisfied identically in ρ. Suppose ρ^{k+2} is linearly independent of the other terms in (3.4). Then we obtain

$$\dddot{\xi} = 0. \tag{3.5}$$

From the results of Lutzky (1978) and Ray and Reid (1979) the equation for ξ leads to the auxiliary equation. Here we restrict ourselves to systems with non-trivial auxiliary equations such as (1.3). The case (3.5) could, of course, be of physical interest, and we exclude it only for brevity. Thus, some of the other terms in (3.4) must be proportional to ρ^{k+2}. There are several possible cases.

First suppose the two terms on the right-hand side of (3.4) are linearly dependent and linearly independent of the terms on the left-hand side. We then obtain from (3.4)

$$\dddot{\xi}/(k+2)^2+2\dot{\xi}\dot{b}/(k+2)+\xi\ddot{b}/(k+2)=0, \tag{3.6}$$

and

$$\xi\sum_i \dot{G}_iF_i+\dot{\xi}\sum_i G_iF_i+\dot{\xi}\sum_i G_i\rho F_i'/(k+2)=0. \tag{3.7}$$

Multiplying (3.6) by ξ we integrate this equation to obtain

$$\xi\ddot{\xi}-\dot{\xi}^2/2+(k+2)\dot{b}\xi^2=2A, \tag{3.8}$$

where A is a constant. After the substitution $\xi=x^2$, this equation reduces to

$$\ddot{x}+(k+2)\dot{b}x/2=A/x^3, \tag{3.9}$$

which is the auxiliary equation for this case. Note that it has the same form as (1.3) for the Lewis invariant. In order to finish the calculation we must solve (3.7). We assume all the F_i's are linearly independent, hence the equation

$$\xi\dot{G}_iF_i+\dot{\xi}G_iF_i+\dot{\xi}G_i\rho F_i'/(k+2)=0 \tag{3.10}$$

holds for each i. Now in these equations F_i and $\rho F_i'$ must be proportional or $\dot{\xi}=0$, which would collapse our auxiliary equation to a trivial form.

Hence

$$\rho F_i'=m_iF_i, \qquad m_i=\text{constants}, \tag{3.11}$$

with the solution

$$F_i=\rho^{m_i}, \qquad m_i\neq k+2. \tag{3.12}$$

Putting this result back into (3.10) yields the solution for G_i,

$$G_i=G_{i0}/x^{[2+2m_i/(k+2)]}, \tag{3.13}$$

where the G_{i0} are arbitrary constants. The Lagrangian for this case can then be written

$$L=\tfrac{1}{2}\rho^k\dot{\rho}^2+b\rho^{k+1}\dot{\rho}+\sum_i G_{i0}\rho^{m_i}/x^{[2+2m_i/(k+2)]}, \tag{3.14}$$

where x satisfies the auxiliary equation (3.9). The invariant in this case takes the form

$$I=\tfrac{1}{2}\rho^k[\dot{\rho}x-2\rho\dot{x}/(k+2)]^2+2A\rho^{k+2}/[(k+2)x]^2-\sum_i G_{i0}\rho^{m_i}/x^{2m_i/(k+2)}. \tag{3.15}$$

The equation of motion is (2.6), with the substitutions $l=k+1$, $F_i=\rho^{m_i}$ and $G_i=G_{i0}/x^{[2+2m_i/(k+2)]}$. The equation of motion (2.6), with these substitutions, is linked to the auxiliary equation (3.9) by the invariant (3.15). Notice that, whereas $G_i(t)$, $F_i(\rho)$ started off in the original Lagrangian as arbitrary functions, the imposition of the symmetry has forced them to have special forms.

The system represented by (3.9), (3.14) and (3.15) still represents many special cases, depending on the values assumed by the various constants. One interesting case is when there is only one non-zero $G_{i0}=G_0$ and $m_i=m=-(k+2)$. In this case the function $G=G_0$ is constant, and the ρ and x equation are uncoupled. In this case the invariant (3.15) can be used to solve the ρ equation in terms of solutions to the x equation and vice versa. One writes out the invariants I_1 and I_2 for two solutions x_1 and x_2, keeping the ρ solution the same. Then one eliminates $\dot\rho$ between the two invariants and obtains the solution for ρ in terms of I_1, I_2, x_1, x_2. In the literature this is referred to as a 'nonlinear superposition' principle (Ames 1978). If the ρ and x equations are coupled, this technique will not work.

As a second example of the solution of (3.4) we assume one of the $F_i = F$ is of the form ρ^{k+2}, and again assume $l+1 = k+2$. In this case we obtain from (3.4)

$$\dddot{\xi}/(k+2)^2 + 2\dot\xi\dot b/(k+2) + \xi\ddot b/(k+2) - \xi\dot G - 2\dot\xi G = 0 \tag{3.16}$$

and

$$\xi \sum_i G_i F_i + \dot\xi \sum_i G_i F_i + \dot\xi \sum_i G_i \rho F_i/(k+2) = 0. \tag{3.17}$$

The integration of (3.16) leads as before to the auxiliary equation

$$\ddot x + (k+2)[\dot b - (k+2)G]x/2 = A/x^3, \tag{3.18}$$

where A is an arbitrary constant. The solution of (3.17) is the same as before and leads to the same results (3.12) and (3.13) for F_i and G_i respectively. The Lagrangian for this case is

$$L = \tfrac{1}{2}\rho^k\dot\rho^2 + b\rho^{k+1}\dot\rho + G\rho^{k+2} + \sum_i G_{i0}\rho^{m_i}/x^{[2+2m_i/(k+2)]}, \qquad m_i \neq k+2. \tag{3.19}$$

Note that in this case the function $G(t)$ is arbitrary and appears in the auxiliary equation, whereas the $G_i(t)$ are determined in terms of the auxiliary equation via (3.15). The invariant for this case would be calculated as before using (2.10). We also have the same special case as considered for the previous example where ρ and x equations decouple, and we obtain 'nonlinear superposition'.

There are several other different solutions to (3.4) which lead to different equations of motion, auxiliary equations and invariants I. For example, assume $l+1 \neq k+2$ and one of the $F_i = F$ is ρ^{k+2}. The Lagrangian for this case would be

$$L = \tfrac{1}{2}\rho^k\dot\rho^2 + b\rho^l\dot\rho + G(t)\rho^{k+2} + \sum_i G_i\rho^{m_i}, \tag{3.20}$$

with $m_i \neq k+2$ or $l+1$. G_i, as before, is found to be (3.13), where x is a solution to the auxiliary equation (3.18) with $b = 0$. The ρ^{l+1} equation leads to an expression for $\dot b$ in terms of x, namely

$$\dot b = B/x^{2[1+(l+1)/(k+2)]}, \tag{3.21}$$

where B is a constant.

We shall leave the working of further examples to the interested reader.

4. Conclusions

In this paper we have applied Noether's theorem to the Lagrangian

$$L = \tfrac{1}{2}\rho^k\dot\rho^2 + b(t)\rho^l\dot\rho + \sum_i G_i(t)F_i(\rho), \tag{4.1}$$

where G_i, F_i, b are initially arbitrary functions. The application of Noether's theorem leads to (3.4) which must be satisfied identically. Solving (3.4) one obtains an equation for x called the auxiliary equation, and then the Noether invariant I in terms of ρ, $\dot{\rho}$, x, $\dot{x}$ and the other functions in the Lagrangian. These results generalise the results obtained by Sarlet (1978) and also obtain the results in a completely different and simpler manner. Of course, Sarlet was interested in more than just finding invariants, he was interested in finding Hamiltonians for which Kruskal's theory works in a closed form.

If the equations of motion and the auxiliary equation are uncoupled, then the invariant leads to a 'nonlinear superposition' of solutions.

The results of this paper, along with Lutzky (1978) and Ray and Reid (1979), show that Noether's theorem is a powerful tool in searching for invariants in time-dependent systems. Application of Noether's theorem to different Lagrangians and a detailed comparison with other methods for generating invariants are topics for further study.

Note added in proof. If in the Lagrangian (2.5) we transform the dependent variable by

$$\rho = y^{2/(k+2)}, \tag{4.2}$$

the Lagrangian takes the form

$$L = \tfrac{1}{2}\dot{y}^2 + by^{(2l-k)/(k+2)}\dot{y} + \sum G_i(t)F_i(y), \tag{4.3}$$

where b, G_i and $F_i(y)$ are still arbitrary functions and we have rescaled L by a constant. Substituting for the second term in (4.3) the expression

$$by^{(2l-k)/(k+2)}\dot{y} = \frac{(k+2)}{2l+2}\frac{\mathrm{d}}{\mathrm{d}t}(by^{(2l+2)/(k+2)}) - \frac{(k+2)}{2l+2}\dot{b}y^{(2l+2)/(k+2)}, \tag{4.4}$$

and calling $G(t) = [(k+2)/(2l+2)]\dot{b}$ the Lagrangian, after dropping the total time derivative in (4.4), becomes

$$L = \tfrac{1}{2}\dot{y}^2 + G(t)y^{(2l+2)/(k+2)} + \sum G_i(t)F_i(y). \tag{4.5}$$

Since G_i and F_i are arbitrary functions the term $G(t)y^{(2l+2)/(k+2)}$ can be included in the sum without loss of generality. The Lagrangian now takes the form

$$L = \tfrac{1}{2}\dot{y}^2 + \sum G_i(t)F_i(y). \tag{4.6}$$

The Lagrangian (4.6) is of the form considered by Ray and Reid (1979), e.g. (1.5). Therefore, the invariants associated with the polynomial Lagrangians considered in this paper are transformable into the form discussed in Ray and Reid (1979).

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